

# Abstract Hermitian Algebras I. Spectral Resolution

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## Abstract

We refer to the real Jordan Banach algebra of bounded Hermitian operators on a Hilbert space as a Hermitian algebra. We define an abstract Hermitian algebra (AH-algebra) to be the directed group of an e-ring that contains a semitransparent element, has the quadratic annihilation property, and satisfies a Vigier condition on pairwise commuting ascending sequences. All of this terminology is explicated in this article, where we launch a study of AH-algebras. Here we establish the fundamental properties of AH-algebras, including the existence of polar decompositions and spectral resolutions, and we show that two elements of an AH-algebra commute if and only if their spectral projections commute. We employ spectral resolutions to assess the structure of maximal pairwise commuting subsets of an AH-algebra.

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# 1 Introduction

We shall refer to the real Banach space  $\mathbb{G}(\mathfrak{H})$  of bounded Hermitian operators on a Hilbert space  $\mathfrak{H}$ , organized in the usual way into a partially ordered real vector space, as the *Hermitian algebra* of  $\mathfrak{H}$ . We call  $\mathbb{G}(\mathfrak{H})$  an “algebra” because it is, in fact, a JW-algebra in the sense of [17, p. 3], and we use the nonstandard notation  $\mathbb{G}(\mathfrak{H})$  because, at least at first, we shall be focusing on its structure as an *partially ordered additive abelian group* [12, p. 1].

Our purpose in this article is to introduce and launch a study of a generalization of  $\mathbb{G}(\mathfrak{H})$  which we call an *abstract Hermitian algebra*, or an *AH-algebra* for short. We derive the basic properties of an AH-algebra, including the existence polar decompositions and of spectral resolutions for each of its elements. In subsequent articles, we shall show that, by analogy with AW\*-algebras and JW-algebras, AH-algebras admit a classification into types I, II, and III, and that an appropriate theory of dimension and symmetries exists for such an algebra. AH-algebras may be regarded as a class of *quantum structures* in the sense of [1].

In the sequel,  $\mathbb{R}$  denotes the ordered field of real numbers and  $\mathbb{N}$  is the set of positive integers. Also,  $\mathfrak{H}$  is a Hilbert space with inner product  $\langle \cdot | \cdot \rangle$ ,  $\mathbb{B}(\mathfrak{H})$  is the Banach \*-algebra with the uniform operator norm  $\| \cdot \|$  of all bounded linear operators on  $\mathfrak{H}$ , and as mentioned above,  $\mathbb{G}(\mathfrak{H})$  is the real Banach space under  $\| \cdot \|$  of all Hermitian operators in  $\mathbb{B}(\mathfrak{H})$ . As usual,  $\mathbb{G}(\mathfrak{H})$  is organized into a partially ordered real linear space by defining  $A \leq B$  for  $A, B \in \mathbb{G}(\mathfrak{H})$  iff  $\langle A\psi | \psi \rangle \leq \langle B\psi | \psi \rangle$  for all  $\psi \in \mathfrak{H}$ . The zero and identity operators on  $\mathfrak{H}$  are denoted by  $\mathbf{0}, \mathbf{1} \in \mathbb{G}(\mathfrak{H})$ , and we denote the “unit interval” in  $\mathbb{G}(\mathfrak{H})$  by  $\mathbb{E}(\mathfrak{H}) := \{E \in \mathbb{G}(\mathfrak{H}) : \mathbf{0} \leq E \leq \mathbf{1}\}$ . Following G. Ludwig [15], operators  $A \in \mathbb{E}(\mathfrak{H})$  are called *effect operators* on  $\mathfrak{H}$ . The complete atomic orthomodular lattice (OML) [14] of all (orthogonal) *projection operators* on  $\mathfrak{H}$  is denoted by  $\mathbb{P}(\mathfrak{H}) := \{P \in \mathbb{G}(\mathfrak{H}) : P = P^2\}$ . We note that

$$\mathbf{0}, \mathbf{1} \in \mathbb{P}(\mathfrak{H}) \subseteq \mathbb{E}(\mathfrak{H}) \subseteq \mathbb{G}(\mathfrak{H}) \subseteq \mathbb{B}(\mathfrak{H}).$$

As we proceed, we shall use  $\mathbb{B}(\mathfrak{H})$ ,  $\mathbb{G}(\mathfrak{H})$ ,  $\mathbb{E}(\mathfrak{H})$ , and  $\mathbb{P}(\mathfrak{H})$  to motivate and illustrate various concepts.

## 2 e-Rings

The following notion of an e-ring was introduced in [7] and further studied in [8, 10] as a generalization of the pair  $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ .

**2.1 Definition.** An *e-ring* is a pair  $(R, E)$  consisting of an associative ring  $R$  with unity 1 and a subset  $E \subseteq R$  of elements called *effects* such that  $0, 1 \in E$ ;  $e \in E \implies 1 - e \in E$ ; and the set  $E^+$  consisting of all finite sums  $e_1 + e_2 + \dots + e_n$  with  $e_1, e_2, \dots, e_n \in E$  satisfies the following conditions: For all  $a, b \in E^+$ ,

- (i)  $-a \in E^+ \implies a = 0$ ,                      (ii)  $1 - a \in E^+ \implies a \in E$ ,
- (iii)  $ab = ba \implies ab \in E^+$ ,                      (iv)  $aba \in E^+$ ,
- (v)  $aba = 0 \implies ab = ba = 0$ , and                      (vi)  $(a - b)^2 \in E^+$ .

If  $(R, E)$  is an e-ring, then the subgroup

$$G := \{a - b : a, b \in E^+\} = E^+ - E^+$$

of the additive group of the ring  $R$  is called the *directed group* of  $(R, E)$ , and  $P := \{p \in G : p = p^2\}$  is called the set of *projections* in  $G$ . The group  $G$  is organized into a partially ordered abelian group with positive cone  $E^+$  by defining, for all  $g, h \in G$ ,  $g \leq h \iff h - g \in E^+$ .

It is not difficult to check that  $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$  is an e-ring, the partially ordered additive abelian group  $\mathbb{G}(\mathfrak{H})$  is its directed group, and  $\mathbb{P}(\mathfrak{H})$  is its set of projections. Fundamental properties of e-rings are developed in [7]. Further examples of e-rings, in addition to the prototype  $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ , as well as motivation for the developments that follow can be found in [7, 8, 10].

**2.2 Standing Assumptions.** *In the sequel, we assume that  $(R, E)$  is an e-ring,  $E^+$  is the set of all sums of finite sequences of effects in  $E$ ,  $E^+$  is the positive cone for the directed group  $G$  of  $(R, E)$ , and  $P$  is the set of projections in  $(R, E)$ . To avoid trivialities, we also assume that  $0 \neq 1$ .*

We note that  $G$  is in fact a *directed group* in the technical sense that it is generated by its own positive cone  $E^+$  [12, p. 4], and the set  $E$  is the *unit interval* in  $G$ , i.e.,  $E = \{e \in G : 0 \leq e \leq 1\}$ . Also, 1 is an *order unit*<sup>1</sup> in  $G$  [12, p. 4], i.e., if  $g \in G$ , there exists  $n \in \mathbb{N}$  such that  $g \leq n \cdot 1$ . Evidently,

$$0, 1 \in P \subseteq E \subseteq E^+ \subseteq G \subseteq R.$$

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<sup>1</sup>Some authors use the terminology *strong order unit*.

We understand that  $E^+$ ,  $E$ , and  $P$  are partially ordered by the restrictions of the partial order  $\leq$  on  $G$ . By [7, Theorem 2.15],  $P$  is an orthomodular poset (OMP) with  $p \mapsto 1 - p$  as the orthocomplementation. Since the mappings  $g \mapsto -g$ ,  $e \mapsto 1 - e$ , and  $p \mapsto 1 - p$  are order-reversing and of period 2 on  $G$ ,  $E$ , and  $P$ , respectively, there is a *duality principle* whereby properties of existing suprema in  $G$ ,  $E$ , or  $P$  are converted to properties of infima and *vice versa*.

In what follows, we focus attention on the directed group  $G$ , the unit interval  $E \subseteq G$ , and the OMP  $P \subseteq E$  of projections—the enveloping ring  $R$  is just a convenient mathematical environment in which to study the triple  $P \subseteq E \subseteq G$ , and the detailed structure of  $R$  will not concern us here.

**2.3 Definition.** Let  $g, h \in G$ . We define  $gCh$  to mean that  $g$  commutes with  $h$ , i.e., that  $gh = hg$ . If  $A \subseteq G$ , we also define  $C(A)$ , called the *commutant of  $A$  in  $G$*  by  $C(A) := \{g \in G \mid gCa, \forall a \in A\}$ . The set  $CC(A) := C(C(A))$  is called the *bicommutant of  $A$  in  $G$* , and if  $g \in CC(h) := CC(\{h\})$ , we say that  $g$  *double commutes* with  $h$ . We also define  $CPC(g) := C(P \cap C(g))$ , so that  $h \in CPC(g)$  iff  $h$  commutes with every projection that commutes with  $g$ .

In contrast to more common usage, e.g. in operator theory, we use the notation  $C(A)$  and  $CC(A)$  only in relation to elements of the directed group  $G$  rather than to arbitrary elements in the enveloping ring  $R$ . By Definition 2.1 (iii), if  $0 \leq g, h \in G$ , then  $gCh \Rightarrow gh = hg \in G$ ; however, unless  $0 \leq g, h$ , we do not assume *a priori* that  $gCh \Rightarrow gh \in G$ .<sup>2</sup> By the spectral theorem, if  $A \in \mathbb{G}(\mathfrak{H})$ , then  $CPC(A) = CC(A)$ ; in general however, even the condition  $CPC(g) \subseteq C(g)$  may fail.

**2.4 Lemma.** *Let  $e, f \in E$ , let  $p \in P$ , and let  $g, h \in G$ . Then:*

- (i) *If  $eCf$ , then  $0 \leq ef \leq e, f \leq 1$  and  $0 \leq e^2 \leq e \leq 1$ .*
- (ii)  *$e \leq p \Leftrightarrow e = ep \Leftrightarrow e = pe$  and  $p \leq e \Leftrightarrow p = pe \Leftrightarrow p = ep$ .*
- (iii)  *$pgp, php \in G$ , and if  $g \leq h$ , then  $pgp \leq php$ .*

*Proof.* For (i) and (ii), see [7, Lemma 2.6, Theorem 2.9, Corollary 2.10]. By [7, Lemma 2.4 (iv)],  $pgp, php \in G$ , and if  $g \leq h$ , then  $0 \leq h - g$ , so  $0 \leq p(h - g)p = php - pgp$  by [7, Lemma 2.4 (v)], and (iii) follows.  $\square$

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<sup>2</sup>See Lemma 2.8 (i) below.

Parts (ii) and (iii) of the following theorem are of interest because they provide conditions *not directly involving multiplication* for an effect to be a projection. See [8, Theorem 3.2] for a proof of the theorem.

**2.5 Theorem.** *If  $p \in E$ , then the following conditions are mutually equivalent: (i)  $p \in P$ . (ii) If  $e, f, e + f \in E$ , then  $e, f \leq p \Rightarrow e + f \leq p$ . (iii) If  $e \in E$  with  $e \leq p, 1 - p$ , then  $e = 0$ . (iv)  $\exists n, m \in \mathbb{N}, n \neq m$  and  $p^n = p^m$ .*

**2.6 Corollary.** *Suppose that  $\emptyset \neq Q \subseteq P$  and that  $Q$  has a supremum (respectively, an infimum)  $p$  in  $G$ . Then  $p \in P$  and  $p$  is the supremum (respectively, the infimum) of  $Q$  in  $P$ .<sup>3</sup>*

*Proof.* By duality it is sufficient to consider the case in which  $p$  is the infimum of  $Q$  in  $G$ . As  $0 \leq q$  for all  $q \in Q$ , we have  $0 \leq p$ . Choose any  $q_0 \in Q$ . Then  $0 \leq p \leq q_0 \leq 1$ , so  $p \in E$ . To prove that  $p \in P$ , suppose that  $e, f, e + f \in E$  with  $e, f \leq p$ . Then, for all  $q \in Q$ , we have  $e, f \leq q$ , whereupon  $e + f \leq q$  by Theorem 2.5 (ii), and it follows that  $e + f \leq p$ , whence  $p \in P$  by Theorem 2.5 (ii) again. As  $p \in P$ , it is clear that  $p$  is the infimum of  $Q$  in  $P$ .  $\square$

As we progress, we shall study conditions on  $G$ ,  $E$ , and  $P$  that are suggested by properties of the prototypes  $\mathbb{G}(\mathfrak{H})$ ,  $\mathbb{E}(\mathfrak{H})$ , and  $\mathbb{P}(\mathfrak{H})$ . Among these are the following.

## 2.7 Definition.

- (i) If there is an effect  $h \in E$  such that  $2h = 1$ , then  $h$  is unique, and we write  $\frac{1}{2} := h$  [8, Definition 4.1]. For reasons elucidated [8, Section 4], we call  $\frac{1}{2}$ , if it exists, the *semitransparent effect*.
- (ii)  $G$  has the *quadratic annihilation (QA) property* iff, for all  $g, h \in G$ ,  $gh^2g = 0 \Rightarrow gh = hg = 0$ .
- (iii)  $G$  is *archimedean* [12, p. 20] iff, whenever  $g, h \in G$  and  $ng \leq h$  for all  $n \in \mathbb{N}$ , it follows that  $g \leq 0$ .

Of course,  $\frac{1}{2}\mathbf{1}$  is the semitransparent effect operator in  $\mathbb{E}(\mathfrak{H})$ . If  $A, B \in \mathbb{G}(\mathfrak{H})$ , then the adjoint of  $BA$  is  $(BA)^* = AB$ , so  $AB^2A = (BA)^*(BA) = \mathbf{0}$  implies that  $AB = BA = \mathbf{0}$ ; i.e.,  $\mathbb{G}(\mathfrak{H})$  has QA. Clearly,  $\mathbb{G}(\mathfrak{H})$  is archimedean.

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<sup>3</sup>In general, the converse of Corollary 2.6 fails, i.e., if  $Q \subseteq P$  and the supremum (respectively, the infimum) of  $Q$  in  $P$  exists, it need not be the supremum (respectively, the infimum) of  $Q$  in  $G$ .

**2.8 Lemma.** Suppose that  $\frac{1}{2} \in E$ , let  $g, h, k \in G$ , and let  $n \in \mathbb{N}$ . Then: (i)  $gCh \Rightarrow gh = hg \in G$ . (ii)  $ghg \in G$ . (iii)  $\frac{1}{2}(gh + hg) \in G$ . (iv)  $g^n \in G$ . (v) If  $G$  is archimedean, then  $g^n = 0 \Rightarrow g = 0$ . (vi) If  $0 \leq k \in C(g) \cap C(h)$ , then  $g \leq h \Rightarrow gk \leq hk$ .

*Proof.* For (i), (ii), (iii), and (iv), see [8, Theorem 4.1]; for (v), see [8, Theorem 4.2].

(vi) Assume the hypotheses. Then  $0 \leq h - g, k$  and  $(h - g)Ck$ , so  $0 \leq (h - g)k = hk - gk$  by Definition 2.1 (iii), and by part (i),  $hk, gk \in G$ .  $\square$

**2.9 Lemma.** Suppose that  $G$  has QA and let  $g, h \in G$ . Then  $gh = 0 \Rightarrow hg = 0$ .

*Proof.* By QA,  $gh = 0 \Rightarrow gh^2g = 0 \Rightarrow gh = hg = 0 \Rightarrow hg = 0$ .  $\square$

### 3 AH-Algebras

We maintain Standing Assumptions 2.2.

#### 3.1 Definition.

- (i)  $G$  has the *Vigier (V) property* [8, Definition 5.1] iff every ascending sequence  $g_1 \leq g_2 \leq \dots$  in  $G$  that is bounded above in  $G$  has a supremum  $g$  in  $G$ , and  $g \in CC(\{g_n : n \in \mathbb{N}\})$ .
- (ii)  $G$  has the *complete Vigier (complete V) property* iff every ascending net  $(g_\alpha)_{\alpha \in A}$  in  $G$  that is bounded above in  $G$  has a supremum  $g$  in  $G$ , and  $g \in CC(\{e_\alpha : \alpha \in A\})$ .
- (iii)  $G$  has the *commutative Vigier (CV) property* iff every ascending sequence  $g_1 \leq g_2 \leq \dots$  of pairwise commuting elements of  $G$  that is bounded above in  $G$  has a supremum  $g$  in  $G$ , and  $g \in CC(\{g_n : n \in \mathbb{N}\})$ .
- (iv) A net  $(g_\alpha)_{\alpha \in A}$  in  $G$  is called a *C-net* iff for all  $\alpha, \beta \in A$ ,  $\alpha \leq \beta \Rightarrow g_\alpha C g_\beta$ . We say that  $G$  has the *complete commutative Vigier (complete CV) property* iff every ascending C-net  $(g_\alpha)_{\alpha \in A}$  in  $G$  that is bounded above in  $G$  has a supremum  $g$  in  $G$ , and  $g \in CC(\{g_\alpha : \alpha \in A\})$ .

An argument originally due to J. Vigier [18] shows that  $\mathbb{G}(\mathfrak{H})$  has the V property [16, page 263]; in fact, by essentially the same argument,  $\mathbb{G}(\mathfrak{H})$  has the complete V property. Obviously, complete  $V \Rightarrow V \Rightarrow CV$  and complete  $V \Rightarrow$  complete  $CV \Rightarrow CV$ .

**3.2 Theorem.** *Suppose that  $\frac{1}{2} \in E$  and  $G$  has the CV property. Then:*

- (i) *If  $0 \leq a \in G$ , then 0 is the infimum in  $G$  of the sequence  $((\frac{1}{2})^n a)_{n \in \mathbb{N}}$ .*
- (ii)  *$G$  is archimedean.*

*Proof.* (i) As  $0 \leq a$ , the sequence  $((\frac{1}{2})^n a)_{n \in \mathbb{N}}$  is descending, bounded below by 0, and its elements commute pairwise, so by CV and duality, it has an infimum  $c$  in  $G$  and  $0 \leq c$ . Also,  $c \leq (\frac{1}{2})^{n+1} a$  for all  $n \in \mathbb{N}$ , whence  $2c \leq (\frac{1}{2})^n a$  for all  $n \in \mathbb{N}$ , so  $2c \leq c$ , i.e.,  $c \leq 0$ , and it follows that  $c = 0$ .

(ii) Suppose  $g, h \in G$  and  $ng \leq h$  for all  $n \in \mathbb{N}$ . As  $G$  is directed, there exist  $a, b \in G$  with  $0 \leq a, b$  and  $h = a - b \leq a$ , whence  $ng \leq a$  for all  $n \in \mathbb{N}$ . In particular,  $2^n g \leq a$  for all  $n \in \mathbb{N}$ , and it follows that  $g \leq (\frac{1}{2})^n a$  for all  $n \in \mathbb{N}$ . Consequently, by part (i),  $g \leq 0$ .  $\square$

Evidently, our prototype  $\mathbb{G}(\mathfrak{H})$  is an AH-algebra as per the following definition.

**3.3 Definition.** The directed group  $G$  of the e-ring  $(R, E)$  is an *abstract Hermitian* (AH) *algebra* iff  $\frac{1}{2} \in E$ ,  $G$  has the quadratic annihilation (QA) property, and  $G$  has the commutative Vigier (CV) property.

**3.4 Standing Assumption.** *Henceforth, we assume that the directed group  $G$  of  $(R, E)$  is an AH-algebra.*

**3.5 Theorem.** *Let  $e \in E$ , let  $d := 1 - e$ , let  $d_1 := \frac{1}{2}d$ , and define the sequence  $(d_n)_{n \in \mathbb{N}}$  recursively by  $d_{n+1} := \frac{1}{2}(d + (d_n)^2)$  for all  $n \in \mathbb{N}$ . Then  $(d_n)_{n \in \mathbb{N}}$  is an ascending sequence of pairwise commuting effects in  $E \cap CC(e)$ , so by CV it has a supremum  $s$  in  $G$  and  $s \in CC(\{d_n : n \in \mathbb{N}\}) \subseteq CC(e)$ . Then  $(1 - s)^2 = e$  with  $1 - s \in CC(e)$ .*

*Proof.* The proof is identical to the proof of [8, Theorem 6.1], which obviously requires only the CV property, not the full V property.  $\square$

As a consequence of Lemma 2.8 (v), Theorem 3.2 (ii), and Theorem 3.5 together with [8, Corollary 6.1 and Theorem 6.4] we have the following.

**3.6 Theorem.** *If  $0 \leq g \in G$ , there exists a unique element in  $G$ , called the square root of  $g$  and denoted by  $g^{1/2}$ , such that  $0 \leq g^{1/2}$  and  $(g^{1/2})^2 = g$ ; moreover,  $g^{1/2} \in CC(g)$ .*

By Definition 2.1 (vi), if  $g = h^2$  for some  $h \in G$ , then  $0 \leq g$ . Conversely, by Theorem 3.6, if  $0 \leq g$ , then there exists  $h \in G$ , namely  $h = g^{1/2}$ , such that  $g = h^2$ . Thus, the positive cone in  $G$  consists precisely of squares of elements of  $G$ .

As usual, we say that an element  $g \in G$  is *invertible* iff there is an element  $h \in G$  such that  $gh = hg = 1$ . If such an  $h$  exists, it is unique, it is called the *inverse* of  $g$ , and it is written as  $g^{-1} := h$ .

**3.7 Theorem.** *Let  $g \in G$  with  $0 \leq g$ . Then  $g$  is invertible iff there exists  $M \in \mathbb{N}$  such that  $1 \leq Mg$ . Moreover, if  $g$  is invertible, then  $0 \leq g^{-1} \in CC(g)$ .*

*Proof.* The proof of [8, Lemma 7.1] goes through as it obviously requires only the CV property rather than the stronger V property.  $\square$

Equipped with the Jordan product  $(A, B) \mapsto \frac{1}{2}(AB + BA)$ , our prototype  $\mathbb{G}(\mathfrak{H})$  is a Jordan algebra. More generally, we have the following result.

**3.8 Theorem.**  *$G$  can be organized into an archimedean partially ordered real vector space and, as such, it is a Jordan algebra with respect to the Jordan product  $(g, h) \mapsto \frac{1}{2}(gh + hg)$ .*

*Proof.* The full V property is not needed for the proof of [8, Theorem 7.2]—only CV is required. Thus,  $G$  can be organized into a partially ordered real vector space that is also a Jordan algebra with the indicated Jordan product, and  $G$  is archimedean by Theorem 3.2 (ii).  $\square$

*In the sequel, we understand that  $G$  is organized into a partially ordered real vector space as per Theorem 3.8. Moreover, we make routine use of Theorems 3.6, 3.7; Lemmas 2.4, 2.8, 2.9; and the following lemma.*

**3.9 Lemma.** *If  $0 \leq g_i \in G$  for  $i = 1, 2, \dots, n$ , there exists  $0 < \lambda \in \mathbb{R}$  such that  $\lambda g_i \in E$  for  $i = 1, 2, \dots, n$ .*

*Proof.* As 1 is an order unit in  $G$ , there exists  $N \in \mathbb{N}$  such that  $g_1, g_2, \dots, g_n \leq N \cdot 1$ . Let  $\lambda := 1/N$ .  $\square$



**3.10 Lemma.** *Let  $g \in G$  be the supremum (respectively, the infimum) in  $G$  of the ascending (respectively, descending) sequence  $(g_n)_{n \in \mathbb{N}} \subseteq G$  of pairwise commuting elements. Suppose  $0 \leq h \in G$  and  $hCg_n$  for all  $n \in \mathbb{N}$ . Then  $gh = hg$  is the supremum (respectively, the infimum) in  $G$  of  $(g_nh)_{n \in \mathbb{N}}$ .*

*Proof.* We prove the lemma for an ascending sequence—the result for a descending sequence then follows by duality. By CV, we have  $gCh$ , so  $gh = hg \in G$ . As  $0 \leq g - g_1$  and  $0 \leq h$ , there exists  $0 \leq \lambda \in \mathbb{R}$  such that  $\lambda(g - g_1), \lambda h \in E$ . For all  $n \in \mathbb{N}$ ,  $0 \leq g - g_n \leq g - g_1$ , so  $0 \leq \lambda(g - g_n) \leq \lambda(g - g_1) \leq 1$ , whence  $\lambda(g - g_n), \lambda h \in E$ . Also,  $\lambda(g - g_n)C\lambda h$ , whence by Lemma 2.4 (i),  $\lambda(g - g_n)\lambda h \leq \lambda(g - g_n)$ , i.e.,

$$\lambda(g - g_n)h \leq g - g_n \text{ for all } n \in \mathbb{N}.$$

As  $g_n \leq g$  and  $0 \leq h \in C(g_n) \cap C(g)$ , Lemma 2.8 (vi) implies that  $g_nh \leq gh$  for all  $n \in \mathbb{N}$ . Suppose  $k \in G$  and  $g_nh \leq k$  for all  $n \in \mathbb{N}$ . We have to show that  $gh \leq k$ . We have

$$\lambda(gh - k) \leq \lambda(gh - g_nh) = \lambda(g - g_n)h \leq g - g_n \text{ for all } n \in \mathbb{N},$$

whence

$$g_n \leq g - \lambda(gh - k) \text{ for all } n \in \mathbb{N},$$

and it follows that  $g \leq g - \lambda(gh - k)$ . Therefore,  $\lambda(gh - k) \leq 0$ , so  $gh - k \leq 0$ , i.e.,  $gh \leq k$ .  $\square$

**3.11 Theorem.** *Let  $g, h \in G$  with  $gCh$  and  $0 \leq g \leq h$ . Then: (i)  $g^2 \leq h^2$  and (ii)  $g^{1/2} \leq h^{1/2}$ .*

*Proof.* (i) Follows from [7, Lemma 2.7 (iii)].

(ii) Choose  $0 < \lambda \in \mathbb{R}$  such that  $e := \lambda g \in E$  and  $f := \lambda h \in E$ . Then  $eCf$ , and  $e \leq f$ . As  $e^{1/2} = \lambda^{1/2}g^{1/2}$  and  $f^{1/2} = \lambda^{1/2}h^{1/2}$ , it will be sufficient to prove that  $e^{1/2} \leq f^{1/2}$ . Define

$$d := 1 - e, \quad c := 1 - f, \quad d_1 := \frac{1}{2}d, \quad c_1 := \frac{1}{2}c$$

and by recursion, for all  $n \in \mathbb{N}$ ,

$$d_{n+1} := \frac{1}{2}(d + (d_n)^2) \quad \text{and} \quad c_{n+1} := \frac{1}{2}(c + (c_n)^2).$$

By Theorem 3.5,  $(d_n)_{n \in \mathbb{N}}$  and  $(c_n)_{n \in \mathbb{N}}$  have suprema  $s$  and  $t$ , respectively, in  $G$ ; moreover,  $e^{1/2} = 1 - s$  and  $f^{1/2} = 1 - t$ . As  $e \leq f$ , we have  $c \leq d$ ,  $c_1 \leq d_1$ , and by part (i) and induction on  $n$ ,  $c_n \leq d_n$  for all  $n \in \mathbb{N}$ . Therefore,  $t \leq s$ , so  $e^{1/2} = 1 - s \leq 1 - t = f^{1/2}$ .  $\square$

## 4 Carrier Projections

We maintain Standing Assumption 3.4

**4.1 Lemma.** *Let  $e \in E$ . Then  $((1 - e)^n)_{n \in \mathbb{N}}$  is a descending sequence of pairwise commuting effects in  $E$ , whence by CV it has an infimum  $q$  in  $G$  and  $q \in CC(e)$ . Moreover,  $1 - q \in CC(e) \cap P$ , and for all  $h \in G$ ,  $eh = 0 \Leftrightarrow (1 - q)h = 0$ .*

*Proof.* As  $1 - e \in E$ , Lemma 2.4 (i) implies that  $((1 - e)^n)_{n \in \mathbb{N}}$  is a descending sequence in  $E$ , and it is obvious that the terms of this sequence commute pairwise; therefore, by CV, it has an infimum  $q$  in  $G$  and  $q \in CC\{(1 - e)^n : n \in \mathbb{N}\} \subseteq CC(e)$ . Thus,  $1 - q \in CC(e)$ . Evidently,  $0 \leq q \leq 1 - e \leq 1$ , whence  $0 \leq q^{1/2} \leq 1$  by Theorem 3.11 (ii), i.e.,  $q^{1/2} \in E$ , and it follows from Lemma 2.4 (i) that  $q = (q^{1/2})^2 \leq q^{1/2}$ . For every  $n \in \mathbb{N}$ , we have  $q \leq (1 - e)^{2n}$ , so by Theorem 3.11 (ii) again,  $q^{1/2} \leq (1 - e)^n$ , and it follows that  $q^{1/2} \leq q$ . Therefore,  $q^{1/2} = q$ , so  $q = q^2 \in P$ , whence  $1 - q \in P \cap CC(e)$ .

Suppose  $h \in G$  and  $eh = 0$ . Then  $0 \leq h^2$  and  $hCe$ , therefore  $h^2C(1 - e)^n$  for all  $n \in \mathbb{N}$ . By Lemma 3.10,  $h^2q = qh^2$  is the infimum in  $G$  of the sequence  $(h^2(1 - e)^n)_{n \in \mathbb{N}}$ . But  $h^2(1 - e) = h^2$ , and by induction on  $n$ ,  $h^2(1 - e)^n = h^2$  for all  $n \in \mathbb{N}$ , so all terms in the sequence  $(h^2(1 - e)^n)_{n \in \mathbb{N}}$  are equal to  $h^2$ , and it follows that  $h^2q = h^2$ . Therefore,  $(1 - q)h^2(1 - q) = 0$ , so  $(1 - q)h = 0$  by QA. Thus,  $eh = 0 \Rightarrow (1 - q)h = 0$ .

Conversely, suppose that  $(1 - q)h = 0$ . As  $q$  is the infimum in  $G$  of  $((1 - e)^n)_{n \in \mathbb{N}}$ , we have  $q \leq 1 - e$ , so  $e \leq 1 - q \in P$ , and it follows that  $e = e(1 - q)$ . Therefore  $eh = e(1 - q)h = 0$ , and we have  $eh = 0 \Leftrightarrow (1 - q)h = 0$ .  $\square$

**4.2 Theorem.** *For each  $g \in G$  there is a uniquely determined projection  $g^\circ \in P$  such that, for all  $h \in G$ ,  $gh = 0 \Leftrightarrow g^\circ h = 0$ . Moreover,  $g^\circ \in P \cap CC(g)$ .*

*Proof.* Let  $g \in G$ . As  $0 \leq g^2$ , there exists  $0 < \lambda \in \mathbb{R}$  such that  $e := \lambda g^2 \in E$ . By Lemma 4.1, there is a projection  $g^\circ \in P \cap CC(e) = CC(g^2) \subseteq CC(g)$  such that, for all  $h \in G$ ,  $eh = 0 \Leftrightarrow g^\circ h = 0$ . By QA, for all  $h \in G$ , we have  $gh = 0 \Rightarrow g^2h = 0 \Rightarrow hg^2h = 0 \Rightarrow gh = 0$ , so

$$gh = 0 \Leftrightarrow g^2h = 0 \Leftrightarrow \lambda g^2h = 0 \Leftrightarrow eh = 0 \Leftrightarrow g^\circ h = 0.$$

To prove uniqueness, suppose  $p \in P$  and  $gh = 0 \Leftrightarrow ph = 0$  for all  $h \in G$ . Then  $g^\circ h = 0 \Leftrightarrow ph = 0$  for all  $h \in G$ . Putting  $h = 1 - p$ , we find that  $g^\circ(1 - p) = 0$ , i.e.,  $g^\circ = g^\circ p$ , so  $g^\circ \leq p$ . By symmetry,  $p \leq g^\circ$ , so  $p = g^\circ$ .  $\square$

**4.3 Definition.** If  $g \in G$ , the uniquely determined projection  $g^\circ$  in Theorem 4.2 is called the *carrier projection* of  $g$ .

As  $\mathbb{G}(\mathfrak{H})$  is an AH-algebra, it follows that each Hermitian operator  $A \in \mathbb{G}(\mathfrak{H})$  has a carrier projection  $A^\circ \in \mathbb{P}(\mathfrak{H}) \cap CC(A)$ . In fact, as is easily seen,  $A^\circ$  is just the projection onto the orthogonal complement of the null space of  $A$ .

In view of Lemma 2.9, the carrier projection  $g^\circ \in P$  of  $g \in G$  is characterized not only by the “right annihilation” condition  $gh = 0 \Leftrightarrow g^\circ h = 0$  for all  $h \in G$ , but also by the corresponding “left annihilation” condition  $hg = 0 \Leftrightarrow hg^\circ = 0$  for all  $h \in G$ . Therefore,  $G$  has the so-called *carrier property* [10, Definition 3.3], and the results of [10, Section 3] are at our disposal.

**4.4 Lemma.** Let  $g, h \in G$ ,  $p \in P$ , and  $e \in E$ . Then: (i)  $g^\circ \leq p \Leftrightarrow gp = pg = g$ . (ii)  $g = g^\circ g = gg^\circ$ . (iii)  $p \leq 1 - g^\circ \Leftrightarrow gp = pg = 0$ . (iv)  $e^\circ$  is the smallest projection  $p \in P$  such that  $e \leq p$ . (v)  $(g^\circ)^\circ = g^\circ$ . (vi)  $gh = 0 \Leftrightarrow g^\circ h^\circ = 0 \Leftrightarrow g^\circ \leq 1 - h^\circ$ .

*Proof.* (i)–(v) follow from [10, Lemma 3.4]. To prove (vi), we observe that  $gh = 0 \Leftrightarrow g^\circ h = 0$  and by the “left annihilation” condition  $g^\circ h = 0 \Leftrightarrow g^\circ h^\circ = 0$ . Moreover, as both  $g^\circ$  and  $h^\circ$  are projections,  $g^\circ h^\circ = 0 \Leftrightarrow g^\circ \leq 1 - h^\circ$ .  $\square$

**4.5 Theorem.**  $P$  is a  $\sigma$ -complete orthomodular lattice (OML). Moreover, if  $G$  has the complete CV property, then  $P$  is a complete OML.

*Proof.* That  $P$  is an OML follows from [10, Theorem 3.5]. Let  $p_1 \leq p_2 \leq \dots$  be an ascending sequence in  $P$ . To prove that  $P$  is  $\sigma$ -complete, it will be sufficient to show that  $(p_n)_{n \in \mathbb{N}}$  has a supremum in  $P$ . By Lemma 2.4 (ii), the projections in the sequence  $(p_n)_{n \in \mathbb{N}}$  commute pairwise, whence by CV,  $(p_n)_{n \in \mathbb{N}}$  has a supremum  $p$  in  $G$ . By Corollary 2.6,  $p \in P$  and  $p$  is the supremum of  $(p_n)_{n \in \mathbb{N}}$  in  $P$ .

Suppose  $G$  has the complete CV-property, let  $Q \subseteq P$ , let  $\mathcal{F}$  be the directed set under inclusion of all finite subsets  $F$  of  $Q$ , and for  $F \in \mathcal{F}$ , let  $q_F$  be the supremum in  $P$  of  $F$ . Then  $((q_F)_{F \in \mathcal{F}})$  is an ascending C-net in  $G$  bounded above by 1, and (arguing as above), one shows that its supremum in  $G$  belongs to  $P$  and is the supremum of  $Q$  in  $P$ .  $\square$

**4.6 Definition.** If  $A \subseteq G$ , then  $p \in P$  is a *carrier projection* for  $A$  iff, for all  $h \in G$ , the condition  $ah = 0$  for all  $a \in A$  is equivalent to the condition

$ph = 0$ . Clearly, if  $A$  has a carrier projection  $p$ , then it is unique, and we shall denote it by  $A^\circ := p$ .

We omit the straightforward proof of the following.

**4.7 Theorem.** *Let  $A \subseteq G$ . Then  $A^\circ$  exists iff  $\{a^\circ : a \in A\}$  has a supremum  $p$  in  $P$ , in which case  $A^\circ = p$ . Therefore,  $P$  is a complete OML iff every subset  $A \subseteq G$  has a carrier projection  $A^\circ$ .*

If  $p, q \in P$ , we denote the supremum and infimum of  $p$  and  $q$  in  $P$  by  $p \vee q$  and  $p \wedge q$ , respectively.

**4.8 Lemma.** *Let  $p, q \in P$ . Then: (i)  $p \leq q \Leftrightarrow q - p \in P$ . (ii) If  $p \leq q$ , then  $q - p = q \wedge (1 - p)$ . (iii) If  $p + q \in P$ , then  $pq = qp = 0$  and  $p + q = p \vee q$ . (iv) If  $pCq$ , then  $p \vee q = p + q - pq$ ,  $p \wedge q = pq$ , and  $p + q = p \vee q + p \wedge q$ .*

*Proof.* For (i) and (ii), see [7, Theorem 2.9 and Corollary 2.14]. For (iii), see [7, Theorem 2.11 and Corollary 2.13]. For (iv), see [7, Theorem 2.12 and Corollary 2.13].  $\square$

**4.9 Definition.** Let  $g \in G$ . As  $0 \leq g^2$ , we can and do define  $|g| := (g^2)^{1/2}$ . Also, we define  $g^+ = \frac{1}{2}(|g| + g)$  and  $g^- = \frac{1}{2}(|g| - g)$ .

**4.10 Lemma.** *Let  $g \in G$  and let  $p := (g^+)^\circ$ . Then:*

- |                              |                                   |
|------------------------------|-----------------------------------|
| (i) $ g ^2 = g^2$ .          | (ii) $ g , g^+, g^- \in CC(g)$ .  |
| (iii) $g = g^+ - g^-$ .      | (iv) $0 \leq  g  = g^+ + g^-$ .   |
| (v) $g^+g^- = g^-g^+ = 0$ .  | (vi) $ -g  =  g $ .               |
| (vii) $g^- = (-g)^+$ .       | (viii) $g^+ = (-g)^-$ .           |
| (ix) $p \in CC(g)$           | (x) $pC g $                       |
| (xi) $pg = g^+$ .            | (xii) $(1 - p)g = -g^-$ .         |
| (xiii) $0 \leq p g  = g^+$ . | (xiv) $0 \leq (1 - p) g  = g^-$ . |

*Proof.* (i)–(viii) are obvious. By Theorem 4.2 and (ii), we have  $p \in CC(g^+) \subseteq CC(g)$ , proving (ix), and (x) follows from (ix) and (ii). We have  $pg^+ = g^+$ , and since  $g^+g^- = 0$ , we also have  $pg^- = 0$ ; hence (xi) and (xii) follow from  $g = g^+ - g^-$ . Likewise,  $p|g| = g^+$  and  $(1 - p)|g| = g^-$  follow from  $|g| = g^+ + g^-$ . Since  $0 \leq |g|, p, 1 - p$ , Definition 2.1 (iii) implies that  $0 \leq p|g| = g^+$  and  $0 \leq (1 - p)|g| = g^-$ , proving (xiii) and (xiv).  $\square$

**4.11 Corollary.** *If  $g \in G$ , then  $g^+$  and  $g^-$  are characterized by the properties  $g = g^+ - g^-$ ,  $g^+g^- = 0$ , and  $0 \leq g^+ + g^-$ .*

*Proof.* Suppose  $a, b \in G$ ,  $g = a - b$ ,  $ab = 0$ , and  $0 \leq a + b$ . Then  $ab = ba = 0$ , whence  $g^2 = a^2 + b^2 = (a + b)^2$ , and as  $0 \leq a + b$ , it follows that  $a + b = (g^2)^{1/2} = |g|$ . Therefore,  $g^+ = \frac{1}{2}(|g| + g) = \frac{1}{2}(a + b + a - b) = a$  and  $g^- = \frac{1}{2}(|g| - g) = \frac{1}{2}(a + b - a + b) = b$ .  $\square$

## 5 The Comparability and Polar Decomposition Properties

We maintain Standing Assumption 3.4.

**5.1 Definition.** Define  $P^\pm(g) := \{p \in P \cap C(g) \cap CPC(g) : (1 - p)g \leq 0 \leq pg\}$ . We say that  $G$  has the *comparability property* [10, Definition 2.7] iff  $P^\pm(g) \neq \emptyset$  for all  $g \in G$ .<sup>4</sup>

**5.2 Theorem.** *If  $g \in G$ , then  $(g^+)^o \in P^\pm(g)$ , hence  $G$  has the comparability property.*

*Proof.* As  $CC(g) \subseteq CPC(g)$ , parts (ix) and (xi)–(xiv) of Lemma 4.10 imply that  $(g^+)^o \in P^\pm(g)$ .  $\square$

In general, there may be more than one projection in  $P^\pm(g)$ , but it can be shown that  $(g^+)^o$  is the smallest such projection [3, Theorem 3.1]. Moreover, no matter which projection  $p \in P^\pm(g)$  is chosen, one always has  $g^+ = pg$  and  $g^- = -(1 - p)g$  [6, Theorem 3.2].

By [4, Corollary 4.6],  $G$  is a so-called *compressible group* [2, Definition 3.3], and since it has the comparability property, it is a so-called *comgroup* [3, Definition 1.1]. Translating [6, Definition 6.1] to our present context, we observe that  $G$  has the *Rickart projection property* iff, for each  $g \in G$ , there exists  $g' \in G$  such that, for all  $p \in P$ ,  $p \leq g' \Leftrightarrow pg = gp = 0$ . By Lemma 4.4 (iii),  $G$  has the Rickart projection property with  $g' := 1 - g^o$  and  $g'' = g^o$  for all  $g \in G$ . Therefore,  $G$  is a so-called *Rickart comgroup* [3, Definition 1.1], whence by changing notation from  $g'$  to  $1 - g^o$  and from  $g''$  to  $g^o$ , we can invoke all the results of [3] and [6].

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<sup>4</sup>In [6, Definition 3.4] the comparability property was called *general comparability* because, for interpolation groups, it is equivalent to the property of the same name [12, Chapter 8].

**5.3 Lemma.** *Let  $g, h \in G$ . Then: (i) If  $h \in CPC(g)$  and  $g \leq h$ , then  $g^+ \leq h^+$ . (ii) If  $0 \leq g \leq h$ , then  $g^\circ \leq h^\circ$ . (iii) If  $h \in CPC(g)$  and  $g \leq h$ , then  $(g^+)^\circ \leq (h^+)^\circ$ . (iv) If  $(g^+)^\circ = 1$ , then  $0 \leq g$ . (v)  $(g^+)^\circ \leq (g^+)^\circ \vee (g^-)^\circ = (g^+)^\circ + (g^-)^\circ = g^\circ$ .*

*Proof.* For (i), see [6, Lemma 4.4 (i)] and for (ii), see [6, Lemma 6.2 (vi)]. Clearly, (iii) follows from (i) and (ii). For (iv), see [6, Theorem 6.5 (v)], and for (v), see [6, Theorem 6.5 (ii)].  $\square$

**5.4 Definition.** An element  $s \in G$  is called a *signum* of  $g$  iff: (i)  $s \in C(g) \cap CPC(g)$ , (ii)  $0 \leq sg = gs \in G$ , (iii)  $g = s^2g$ , and (iv)  $\forall h \in G, gh = 0 \implies sh = 0$ . We say that  $G$  has the *polar decomposition (PD) property* [10, Definition 4.3] iff every  $g \in G$  has a signum  $s \in G$ .

**5.5 Theorem.** *Let  $g \in G$ . Then  $s := (g^+)^\circ - (g^-)^\circ$  is the unique signum of  $g$ ; hence  $G$  has the polar decomposition (PD) property. Moreover: (i)  $s \in CC(g)$ . (ii)  $g^\circ = s^2$ . (iii)  $|g| = sg = gs$ . (iv)  $g$  has the “polar decomposition”  $g = s|g| = |g|s$ . (v)  $|g|^\circ = g^\circ$ .*

*Proof.* As  $G$  has both the carrier and comparability properties, [10, Theorem 4.10] implies that the signum  $s$  of  $g$  exists,  $s$  is uniquely determined by  $g$ , and  $s = (g^+)^\circ - (g^-)^\circ$ . By Lemma 4.10 (ix),  $(g^+)^\circ \in CC(g)$ . Likewise, by Lemma 4.10  $(g^-)^\circ = ((-g)^+)^\circ \in CC(-g) = CC(g)$ , and (i) follows. See [10, Lemma 4.4 and Theorem 4.7 (iii)] for proofs of (ii), (iii), and (iv). To prove (v), we note that  $gh = 0 \implies sgh = 0 \implies |g|h = 0 \implies s|g|h = 0 \implies gh = 0$ , so  $gh = 0 \Leftrightarrow |g|h = 0$ .  $\square$

**5.6 Theorem.** *Let  $g \in G$ . Then the following conditions are mutually equivalent: (i)  $g$  is invertible. (ii)  $|g|$  is invertible. (iii) There exists  $0 < \lambda \in \mathbb{R}$  such that  $\lambda \cdot 1 \leq |g|$ . Moreover, if  $g^{-1}$  exists, then  $g^{-1} \in CC(g)$  and the signum  $s$  of  $g$  satisfies  $s^2 = 1$ .*

*Proof.* Let  $s$  be the signum of  $g$ . As  $s \in CC(g)$  and  $|g| \in CC(g)$ , the desired equivalences follow from Theorem 3.7 and the obvious facts that if  $g^{-1}$  exists, then  $|g|^{-1} = sg^{-1}$ , and if  $|g|^{-1}$  exists, then  $g^{-1} = s|g|^{-1}$ . Also, if  $g^{-1}$  exists, it is clear that if  $h \in G$ , then  $gh = 0 \Leftrightarrow h = 0$ , so  $g^\circ = 1$ , and therefore  $s^2 = g^\circ = 1$  by Theorem 5.5 (ii).  $\square$

## 6 States and the 1-Norm

We maintain Standing Assumption 3.4.

**6.1 Definition.** If we regard  $G$  and  $\mathbb{R}$  as a ordered additive abelian groups, then an order-preserving group homomorphism  $\omega: G \rightarrow \mathbb{R}$  such that  $\omega(1) = 1$  is called a *state* for  $G$  [12, p. 60]. We denote the set of all states for  $G$  by  $\Omega(G)$ , or simply as  $\Omega$  if  $G$  is understood.

Note that  $\Omega$  is a convex subset of the locally convex real linear topological space  $\mathbb{R}^G$  of real-valued functions on  $G$  with the topology of pointwise convergence. Equipped with the relative topology inherited from  $\mathbb{R}^G$ ,  $\Omega$  is a nonempty compact set [12, Corollary 4.4 and Proposition 6.5] called the *state space* of  $G$ . By [12, Lemma 6.7], every state  $\omega \in \Omega$  is a linear functional on the real linear space  $G$ .

**6.2 Theorem.**  $\Omega$  is “order determining” in the sense that, for  $g, h \in G$ ,  $g \leq h \Leftrightarrow \omega(g) \leq \omega(h)$  for all  $\omega \in \Omega$ .

*Proof.* As  $G$  is archimedean, [12, Theorem 4.14] implies that  $0 \leq h - g \Leftrightarrow 0 \leq \omega(h - g) = \omega(h) - \omega(g)$  for all  $\omega \in \Omega$ .  $\square$

**6.3 Definition.** Define the 1-norm  $\|\cdot\|: G \rightarrow \mathbb{R}^+$  by

$$\|g\| := \inf\{\lambda \in \mathbb{R} : 0 \leq \lambda \text{ and } -\lambda \cdot 1 \leq g \leq \lambda \cdot 1\}$$

for all  $g \in G$ .

**6.4 Theorem.** The 1-norm  $\|\cdot\|$  is a norm on the real linear space  $G$ . Moreover, for all  $g, h \in G$ : (i)  $\|g\| = \max\{|\omega(g)| : \omega \in \Omega\}$ . (ii)  $-h \leq g \leq h \Rightarrow \|g\| \leq \|h\|$ . (iii)  $0 \neq p \Rightarrow \|p\| = 1$ . (iv)  $\|pgp\| \leq \|g\|$ . (v) If  $\beta_i, \beta \in \mathbb{R}$ ,  $0 \leq \beta_i \leq \beta$ , and  $0 \leq u_i \in G$  for all  $i = 1, 2, \dots, n$ , then  $\|\sum_{i=1}^n \beta_i u_i\| \leq \beta \|\sum_{i=1}^n u_i\|$ .

*Proof.* That  $\|\cdot\|$  is a norm on  $G$  as well as properties (i) and (ii) can be deduced from the results in [12, pp. 120–121]. For (iii) and (iv), see [3, Theorem 3.3 (viii), (ix)]. Let  $0 \leq \lambda \in \mathbb{R}$ . By the hypotheses of (v),  $-\beta \sum_{i=1}^n u_i \leq 0 \leq \sum_{i=1}^n \beta_i u_i \leq \beta \sum_{i=1}^n u_i$ , whence (v) follows from (ii).  $\square$

As is well-known, for the archimedean directed group  $\mathbb{G}(\mathfrak{H})$ , the 1-norm coincides with the uniform operator norm.

**6.5 Theorem.** Let  $g, h \in G$ . Then: (i)  $-1 \leq g \leq 1 \Leftrightarrow g^2 \leq 1$ . (ii)  $\|g^2\| = \|g\|^2$ . (iii)  $h = |g| \Rightarrow \|h\| = \|g\|$ . (iv)  $gCh \Rightarrow \|gh\| \leq \|g\|\|h\|$ .

*Proof.* If  $-1 \leq g \leq 1$ , then  $0 \leq 1 - g, 1 + g$  with  $(1 - g)C(1 + g)$ , whence  $0 \leq (1 - g)(1 + g) = 1 - g^2$ , i.e.,  $g^2 \leq 1$ . Conversely,  $g^2 \leq 1 \Rightarrow -1 \leq g \leq 1$  follows from [8, Lemma 4.3 (iii)], proving (i). If  $0 < \lambda \in \mathbb{R}$ , then by replacing  $g$  by  $\lambda^{-1}g$  in (i), we deduce that  $-\lambda \cdot 1 \leq g \leq \lambda \cdot 1 \Leftrightarrow g^2 \leq \lambda^2$ , from which (ii) follows. If  $h = |g|$ , then  $h^2 = g^2$ , so  $\|h\|^2 = \|g\|^2$  by (ii), and (iii) follows. To prove (iv), suppose that  $gCh$ . Then  $|g|C|h|$ , so by (iii) we can assume without loss of generality that  $0 \leq g, h$ . Moreover, we can assume that  $g, h \neq 0$ , so that  $\|g\|, \|h\| \neq 0$ , and define  $e := \|g\|^{-1}g$  and  $f := \|h\|^{-1}h$ . Then  $e, f \in E$  with  $ef = fe$ , hence  $0 \leq ef \leq 1$  by Lemma 2.4 (i), and it follows that  $\|ef\| \leq 1$ . Therefore,  $\|gh\| = \|g\|\|h\|\|ef\| \leq \|g\|\|h\|$ .  $\square$

Recall that  $G$  is said to be *monotone  $\sigma$ -complete* iff every ascending sequence in  $G$  that is bounded above in  $G$  has a supremum in  $G$  [12, Chapter 16]. Thus, if  $G$  has property  $V$ , it is monotone  $\sigma$ -complete.

**6.6 Theorem.** If  $G$  is monotone  $\sigma$ -complete, then it is a real Banach space under the 1-norm.

*Proof.* See [13, Proposition 3.9].  $\square$

**6.7 Theorem.** Let  $(g_n)_{n \in \mathbb{N}}$  be a sequence in  $G$  and let  $g \in G$ . Then:

- (i) If  $g_n \rightarrow g$  in the 1-norm, then for each  $\omega \in \Omega$  we have  $\omega(g_n) \rightarrow \omega(g)$  in  $\mathbb{R}$ .
- (ii) If  $g_1 \leq g_2 \leq \dots$  and  $\omega(g_n) \rightarrow \omega(g)$  in  $\mathbb{R}$  for each  $\omega \in \Omega$ , then  $g$  is the supremum in  $G$  of  $(g_n)_{n \in \mathbb{N}}$ .
- (iii) If  $g_1 \leq g_2 \leq \dots$  is an ascending sequence of pairwise commuting elements in  $G$ ,  $g \in G$ , and  $g_n \rightarrow g \in G$  in the 1-norm, then  $g$  is the supremum in  $G$  of  $(g_n)_{n \in \mathbb{N}}$  and  $g \in CC(\{g_n : n \in \mathbb{N}\})$ .

*Proof.* (i) Suppose that  $g_n \rightarrow g \in G$  in the 1-norm and let  $\omega \in \Omega$ . Let  $\epsilon \in \mathbb{R}$  with  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ ,  $n \geq N \Rightarrow \|g_n - g\| \leq \epsilon$ . By Theorem 6.4 (i), If  $n \geq N$ , then  $|\omega(g_n) - \omega(g)| = |\omega(g_n - g)| \leq \|g_n - g\| \leq \epsilon$ , whence  $\omega(g_n) \rightarrow \omega(g)$ .

(ii) Assume the hypotheses of (ii) and let  $\omega \in \Omega$ . Then  $\omega(g_1) \leq \omega(g_2) \leq \dots$ , and it follows that  $\omega(g)$  is the supremum in  $\mathbb{R}$  of the sequence  $(\omega(g_n))_{n \in \mathbb{N}}$ .



In particular, for each  $n \in \mathbb{N}$ ,  $\omega(g_n) \leq \omega(g)$ , and since  $\omega \in \Omega$  is arbitrary, it follows from Theorem 6.2 that  $g_n \leq g$ . To prove that  $g$  is the supremum in  $G$  of  $(g_n)_{n \in \mathbb{N}}$ , suppose  $h \in G$  and  $g_n \leq h$  for all  $n \in \mathbb{N}$ . Then, for each  $\omega \in \Omega$ , we have  $\omega(g_n) \leq \omega(h)$  for all  $n \in \mathbb{N}$ , whence,  $\omega(g) \leq \omega(h)$ , and since  $\omega \in \Omega$  is arbitrary, it follows that  $g \leq h$ .

(iii) Follows from (i), (ii), and CV.  $\square$

## 7 Spectral Resolution

We maintain Standing Assumption 3.4 and we denote the state space of  $G$  by  $\Omega$ .

**7.1 Definition.** If  $g \in G$ , then the *spectral lower and upper bounds* for  $g$  are defined by  $L_g := \sup\{\lambda \in \mathbb{R} : \lambda \cdot 1 \leq g\}$  and  $U_g := \inf\{\lambda \in \mathbb{R} : g \leq \lambda \cdot 1\}$ , respectively.

**7.2 Theorem.** If  $g \in G$ , then: (i)  $-\infty < L_g \leq U_g < \infty$ . (ii)  $\{\omega(g) : \omega \in \Omega\}$  is the closed interval  $[L_g, U_g] \subseteq \mathbb{R}$ . (iii)  $\|g\| = \max\{|L_g|, |U_g|\}$ . (iv)  $L_{-g} = -U_g$  and  $U_{-g} = -L_g$ .

*Proof.* Parts (i) and (ii) follow as in the proof of [12, Proposition 4.7], (iii) follows as in the proof of [12, Proposition 4.7], and (iv) is obvious.  $\square$

In [3, Section 4], we proved that each element  $g$  in a Rickart comgroup has a rational spectral resolution  $(p_{g,\lambda})_{\lambda \in \mathbb{Q}}$ . Under our current stronger hypotheses, we can extend the rational spectral resolution as follows to obtain a real spectral resolution  $(p_{g,\lambda})_{\lambda \in \mathbb{R}}$  for each element  $g \in G$ .

**7.3 Definition.** Let  $g \in G$  and  $\lambda \in \mathbb{R}$ . We define

$$p_{g,\lambda} := 1 - ((g - \lambda \cdot 1)^+)^{\circ} \in P \text{ and } d_{g,\lambda} := 1 - (g - \lambda \cdot 1)^{\circ} \in P.$$

The family of projections  $(p_{g,\lambda})_{\lambda \in \mathbb{R}}$  is called the *spectral resolution* for  $g$ , and for  $\lambda \in \mathbb{R}$ ,  $d_{g,\lambda}$  is called the  $\lambda$ -*eigenprojection* for  $g$ . If  $d_{g,\lambda} \neq 0$ , then  $\lambda$  is an *eigenvalue* of  $g$ . If  $g$  is understood, we write the spectral resolution for  $g$  as  $(p_{\lambda})_{\lambda \in \mathbb{R}}$  and we write the family of eigenprojections for  $g$  as  $(d_{\lambda})_{\lambda \in \mathbb{R}}$ .

**7.4 Standing Assumptions.** In what follows,  $g \in G$ ;  $L := L_g$  and  $U := U_g$  are the spectral bounds for  $g$ ;  $(p_{\lambda})_{\lambda \in \mathbb{R}}$  is the spectral resolution of  $g$ ; and  $(d_{\lambda})_{\lambda \in \mathbb{R}}$  is the family of eigenprojections for  $g$ .

**7.5 Lemma.** *Let  $(q_\lambda)_{\lambda \in \mathbb{R}}$  be the spectral resolution of  $-g$  and let  $(c_\lambda)_{\lambda \in \mathbb{R}}$  be the family of eigenprojections for  $-g$ . Then, for all  $\lambda \in \mathbb{R}$ , (i)  $q_\lambda = (1 - p_{-\lambda}) + d_{-\lambda} = (1 - p_{-\lambda}) \vee d_{-\lambda}$  and (ii)  $c_\lambda = d_{-\lambda}$ .*

*Proof.* By Lemma 4.10 (vii), we have

$$1 - q_\lambda = ((-g - \lambda \cdot 1)^+)^o = ((-(g - (-\lambda)1))^+)^o = ((g - (-\lambda)1)^-)^o.$$

Thus, by Lemma 5.3 (v),

$$\begin{aligned} (1 - p_{-\lambda}) + (1 - q_\lambda) &= ((g - (-\lambda)1)^+)^o + ((g - (-\lambda)1)^-)^o \\ &= (g - (-\lambda)1)^o = 1 - d_{-\lambda}, \end{aligned}$$

whence, by Lemma 4.8 (iii),  $q_\lambda = (1 - p_{-\lambda}) + d_{-\lambda} = (1 - p_{-\lambda}) \vee d_{-\lambda}$ , proving (i). Finally, it is clear that  $(-h)^o = h^o$  for all  $h \in G$ , so

$$c_\lambda = 1 - (-g - \lambda \cdot 1)^o = 1 - (-(g - (-\lambda)1))^o = 1 - (g - (-\lambda)1)^o = d_{-\lambda}.$$

□

**7.6 Theorem.** *For all  $\lambda, \mu \in \mathbb{R}$ :*

- (i)  $p_\lambda, d_\lambda \in P \cap CC(g)$  and  $d_\lambda C p_\lambda$ .
- (ii)  $p_\lambda g - \lambda p_\lambda \leq 0 \leq (1 - p_\lambda)g - \lambda(1 - p_\lambda)$ .
- (iii)  $\lambda \leq \mu \Rightarrow p_\lambda \leq p_\mu$  and  $p_\mu - p_\lambda = p_\mu \wedge (1 - p_\lambda)$ .
- (iv)  $\lambda < \mu \Rightarrow d_\lambda \leq p_\lambda \leq 1 - d_\mu$ .
- (v)  $\lambda > U \Rightarrow p_\lambda = 1$ , and  $\lambda < U \Rightarrow p_\lambda < 1$ .
- (vi)  $\lambda < L \Rightarrow p_\lambda = 0$ , and  $L < \lambda \Rightarrow 0 < p_\lambda$ .
- (vii)  $L = \sup\{\lambda \in \mathbb{R} : p_\lambda = 0\}$ , and  $U = \inf\{\lambda \in \mathbb{R} : p_\lambda = 1\}$ .
- (viii) If  $\lambda \leq \mu$  and  $q \in P$  with  $q \leq p_\mu - p_\lambda$ , then  $\lambda q \leq qgq \leq \mu q$ .

*Proof.* (i) Clearly,  $C(g - \lambda \cdot 1) = C(g)$  and  $CC(g - \lambda \cdot 1) = CC(g)$ , whence  $p_\lambda, d_\lambda \in P \cap CC(g)$  by Lemma 4.10 (ix) and Theorem 4.2.

(ii) By Theorem 5.2,  $1 - p_\lambda = ((g - \lambda \cdot 1)^+)^o \in P^\pm(g - \lambda \cdot 1)$ , and (ii) then follows from the definition of  $P^\pm(g - \lambda \cdot 1)$

(iii) Assume that  $\lambda \leq \mu$ . Then  $g - \mu \cdot 1 \leq g - \lambda \cdot 1$ , and  $g - \mu \cdot 1 \in CC(g - \lambda \cdot 1)$ ; hence  $p_\lambda \leq p_\mu$  follows from Lemma 5.3 (iii). Thus,  $p_\mu - p_\lambda = p_\mu \wedge (1 - p_\lambda)$  by Lemma 4.8 (ii).

(iv) By Lemma 5.3 (v), we have  $1 - p_\lambda = ((g - \lambda \cdot 1)^+)^o \leq (g - \lambda \cdot 1)^o = 1 - d_\lambda$ , whence  $d_\lambda \leq p_\lambda$ . Assume that  $\lambda < \mu$ . By (i),  $d_\mu \in CC(g)$  and  $p_\lambda \in C(g)$ , so  $d_\mu C p_\lambda$ . By (ii),  $g p_\lambda = p_\lambda g \leq \lambda p_\lambda$ , and as the projection  $d_\mu$  commutes with both  $g$  and  $p_\lambda$ , Lemma 2.8 (vi) implies that

$$d_\mu g p_\lambda \leq d_\mu (\lambda p_\lambda) = \lambda d_\mu p_\lambda.$$

As  $d_\mu = 1 - (g - \mu \cdot 1)^o$ , we have  $(g - \mu \cdot 1) d_\mu = 0$ , i.e.,  $\mu d_\mu = g d_\mu = d_\mu g$ . Therefore,

$$\mu d_\mu p_\lambda = d_\mu g p_\lambda \leq \lambda d_\mu p_\lambda \leq \mu d_\mu p_\lambda,$$

whence  $\mu d_\mu p_\lambda = \lambda d_\mu p_\lambda$ , i.e.,  $(\mu - \lambda) d_\mu p_\lambda = 0$ . But,  $\mu - \lambda > 0$ , so  $d_\mu p_\lambda = 0$ , and it follows that  $p_\lambda \leq 1 - d_\mu$ .

(v) If  $\lambda > U$ , there exists  $\mu \in \mathbb{R}$  such that  $\mu < \lambda$  and  $g \leq \mu \cdot 1 \leq \lambda \cdot 1$ , whereupon  $g - \lambda \cdot 1 \leq 0$ , i.e.,  $(g - \lambda \cdot 1)^+ = 0$ , so  $((g - \lambda \cdot 1)^+)^o = 0$ , and it follows that  $p_\lambda = 1$ . Conversely, if  $p_\lambda = 1$ , then  $((g - \lambda \cdot 1)^+)^o = 0$ , so  $(g - \lambda \cdot 1)^+ = 0$ , whence  $g - \lambda \cdot 1 \leq 0$ , and it follows that  $U \leq \lambda$ ; consequently,  $\lambda < U \Rightarrow p_\lambda < 1$ .

(vi) Suppose  $\lambda < L$ . Then there exists  $\mu \in \mathbb{R}$  such that  $\lambda < \mu$  and  $\mu \cdot 1 \leq g$ . Therefore,  $1 \leq (\mu - \lambda)1 = \mu \cdot 1 - \lambda \cdot 1 \leq g - \lambda \cdot 1 = (g - \lambda \cdot 1)^+$ , and it follows from Lemma 5.3 (ii) that  $1 = 1^o \leq ((g - \lambda \cdot 1)^+)^o = 1 - p_\lambda$ , whence  $p_\lambda = 0$ . Conversely, if  $p_\lambda = 0$ , then  $((g - \lambda \cdot 1)^+)^o = 1$ , whence  $0 \leq (g - \lambda \cdot 1)$ , i.e.,  $\lambda \cdot 1 \leq g$ , by Lemma 5.3 (iv), whereupon  $\lambda \leq L$ ; consequently,  $L < \lambda \Rightarrow 0 < p_\lambda$ .

(vii) Follows directly from (v) and (vi).

(viii) Assume the hypotheses. By (iii),  $q \leq p_\mu$  and  $q \leq 1 - p_\lambda$ ; hence  $q = q p_\mu = p_\mu q$  and  $q = q(1 - p_\lambda) = (1 - p_\lambda)q$  by Lemma 4.8 (i). Also, by (ii),

$$\lambda(1 - p_\lambda) \leq (1 - p_\lambda)g \quad \text{and} \quad p_\mu g \leq \mu p_\mu;$$

hence, by Lemma 2.4 (iii),

$$\lambda q = q \lambda (1 - p_\lambda) q \leq q (1 - p_\lambda) g q = q g q \quad \text{and}$$

$$q g q = q p_\mu g q \leq q \mu p_\mu q = \mu q.$$

Consequently,  $\lambda q \leq q g q \leq \mu q$ . □

**7.7 Theorem.** Suppose that  $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}$  with

$$\lambda_0 < L < \lambda_1 < \dots < \lambda_{n-1} < U < \lambda_n$$

and let  $\gamma_i \in \mathbb{R}$  with  $\lambda_{i-1} \leq \gamma_i \leq \lambda_i$  for  $i = 1, 2, \dots, n$ . Define  $u_i := p_{\lambda_i} - p_{\lambda_{i-1}}$  for  $i = 1, 2, \dots, n$ , and let  $\epsilon := \max\{\lambda_i - \lambda_{i-1} : i = 1, 2, \dots, n\}$ . Then:

$$u_1, u_2, \dots, u_n \in P \cap CC(g), \quad \sum_{i=1}^n u_i = 1, \quad \text{and} \quad \|g - \sum_{i=1}^n \gamma_i u_i\| \leq \epsilon.$$

*Proof.* In the proof, we understand that  $i = 1, 2, \dots, n$  and that all sums are from  $i = 1$  to  $i = n$ . By parts (i) and (iii) of Theorem 7.6, we have  $p_{\lambda_{i-1}} \leq p_{\lambda_i}$  with  $p_{\lambda_{i-1}}, p_{\lambda_i} \in P \cap CC(g)$ , whence  $u_i \in P \cap CC(g)$ . That  $\sum u_i = 1$  follows from parts (v) and (vi) of Theorem 7.6. Since  $u_i \in C(g)$ , Theorem 7.6 (viii) implies that  $\lambda_{i-1}u_i \leq u_i g \leq \lambda_i u_i$  and, adding these inequalities, we find that  $\sum \lambda_{i-1}u_i \leq \sum u_i g = 1 \cdot g = g \leq \sum \lambda_i u_i$ . The latter inequalities together with  $\lambda_{i-1} \leq \gamma_i \leq \lambda_i$  and  $0 \leq u_i$  imply that

$$\begin{aligned} -\sum (\lambda_i - \lambda_{i-1})u_i &\leq -\sum (\gamma_i - \lambda_{i-1})u_i \leq g - \sum \gamma_i u_i \\ &\leq \sum (\lambda_i - \gamma_i)u_i \leq \sum (\lambda_i - \lambda_{i-1})u_i, \end{aligned}$$

whence

$$\|g - \sum \gamma_i u_i\| \leq \|\sum (\lambda_i - \lambda_{i-1})u_i\| \leq \epsilon \|\sum u_i\| = \epsilon \cdot 1 = \epsilon$$

parts (ii) and (v) of Theorem 6.4 and part (iv) of Theorem 6.4 with  $p = 1$ .  $\square$

**7.8 Theorem.** If  $h \in G$ , then  $hCg \Leftrightarrow hCp_\lambda$  for all  $\lambda \in \mathbb{R}$ .

*Proof.* If  $hCg$  and  $\lambda \in \mathbb{R}$ , then  $hCp_\lambda$  by Theorem 7.6 (i). Conversely, suppose that  $hCp_\lambda$  for all  $\lambda \in \mathbb{R}$ . Choose and fix  $\alpha, \beta \in \mathbb{R}$  with  $\alpha < L$  and  $\beta > U$ . As usual, a partition of the closed interval  $[\alpha, \beta] \subseteq \mathbb{R}$  is understood to be a finite sequence  $\Lambda = (\lambda_i)_{i=0,1,2,\dots,n} \subseteq [\alpha, \beta]$  such that  $\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = \beta$ . The closed interval  $[\lambda_{i-1}, \lambda_i]$  is called the  $i$ th subinterval of  $\Lambda$  for  $i = 1, 2, \dots, n$ , and we define  $\epsilon(\Lambda) := \max\{\lambda_i - \lambda_{i-1} : i = 1, 2, \dots, n\}$ . For the partition  $\Lambda$ , we also define  $g(\Lambda) := \sum_{i=1}^n \lambda_{i-1}(p_{\lambda_i} - p_{\lambda_{i-1}})$ , and we have  $\|g - g(\Lambda)\| \leq \epsilon(\Lambda)$  by Theorem 7.7. As  $hCp_\lambda$  for all  $\lambda \in \mathbb{R}$ , we have  $hCg(\Lambda)$ .

By recursion, we define a sequence  $(\Lambda_n)_{n \in \mathbb{N}}$  of partitions of  $[\alpha, \beta]$  as follows:  $\Lambda_1$  is the partition  $\alpha = \lambda_0 < \lambda_1 = \beta$  having only one subinterval,

namely  $[\alpha, \beta]$  itself. From each partition  $\Lambda_n$ , we form the refined partition  $\Lambda_{n+1}$ , with twice as many subintervals as  $\Lambda_n$ , by appending to the partition  $\Lambda_n$  the midpoints of all its subintervals. It is clear that  $g(\Lambda_1) \leq g(\Lambda_2) \leq \dots$  and that  $g(\Lambda_i)Cg(\Lambda_j)$  for all  $i, j \in \mathbb{N}$ . Obviously,  $\epsilon(\Lambda_n) = (\beta - \alpha)/2^{n-1}$ , whence by Theorem 7.7,  $g(\Lambda_n) \rightarrow g$  in the 1-norm  $\|\cdot\|$ . Therefore, by Theorem 6.7 (iii),  $g$  is the supremum of the ascending sequence  $(g(\Lambda_n))_{n \in \mathbb{N}}$  and  $g \in CC(\{g(\Lambda_n) : n \in \mathbb{N}\})$ ; hence  $gCh$ .  $\square$

**7.9 Corollary.** *Let  $g, h \in G$  and let  $A \subseteq G$ . Then: (i)  $gCh$  iff every projection in the spectral resolution of  $g$  commutes with every projection in the spectral resolution of  $h$ . (ii)  $C(C(A) \cap P) = CC(A)$ . (iii)  $CPC(g) = CC(g)$ .*

*Proof.* (i) Follows from Theorem 7.8. As  $C(A) \cap P \subseteq C(A)$ , we have  $CC(A) \subseteq C(C(A) \cap P)$ . Conversely, suppose  $g \in C(C(A) \cap P)$ ,  $h \in C(A)$ , and  $(p_{h,\lambda})_{\lambda \in \mathbb{R}}$  is the spectral resolution of  $h$ . Then by Theorem 7.8,  $p_{h,\lambda} \in C(A) \cap P$ , so  $gCp_{h,\lambda}$  for every  $\lambda \in \mathbb{R}$ , and therefore  $gCh$ . Consequently,  $C(C(A) \cap P) \subseteq CC(A)$ , and (ii) holds. Putting  $A := \{g\}$  in (ii), we obtain (iii).  $\square$

The following theorem indicates the sense in which the spectral resolution of  $g$  is “continuous from the right.”

**7.10 Theorem.** *If  $\alpha \in \mathbb{R}$ , then  $p_\alpha$  is the infimum in the OML  $P$  of  $A := \{p_\mu : \alpha < \mu \in \mathbb{R}\}$ .*

*Proof.* By Theorem 7.6 (iii),  $p_\alpha$  is a lower bound for  $A$ . Suppose that  $r \in P$  is another lower bound for  $A$ . We have to prove that  $r \leq p_\alpha$ . Evidently,  $p_\alpha \vee r$  is a lower bound for  $A$ . Define  $q := (p_\alpha \vee r) - p_\alpha = (p_\alpha \vee r) \wedge (1 - p_\alpha)$  (Lemma 4.8 (ii)). It will be sufficient to prove that  $q = 0$ . Let  $\lambda \in \mathbb{R}$ . If  $\lambda \leq \alpha$ , then  $p_\lambda \leq p_\alpha \leq p_\alpha \vee r$ , so  $p_\lambda Cq$  by Lemma 2.4 (ii). If  $\alpha < \lambda$ , then  $p_\lambda \in A$ , so  $q \leq p_\alpha \vee r \leq p_\lambda$ , and again  $p_\lambda Cq$ ; hence  $gCq$  by Theorem 7.8.

Now suppose that  $\lambda < \mu \in \mathbb{R}$ . Then  $p_\mu \in A$ , so  $q \leq p_\alpha \vee r \leq p_\mu$  and  $q \leq 1 - p_\alpha$ , so  $q \leq p_\mu \wedge (1 - p_\alpha) = p_\mu - p_\alpha$ , and it follows from Theorem 7.6 (viii) that  $\alpha q \leq qgq = gq = qg \leq \mu q$ . Let  $\omega \in \Omega$ . As  $\omega$  is a linear functional on  $G$ , we have

$$\alpha\omega(q) = \omega(\alpha q) \leq \omega(qg) \leq \omega(\mu q) = \mu\omega(q),$$

and since  $\mu > \alpha$  is arbitrary, it follows that  $\omega(\alpha q) = \omega(qg)$ . By Theorem 6.2, we conclude that  $\alpha q = qg = gq$ . Therefore,  $q(g - \alpha \cdot 1) = 0$ , whence  $q \leq 1 - (g - \alpha \cdot 1)^\circ = d_\alpha \leq p_\alpha$  by Theorem 7.6 (iv). But  $q \leq 1 - p_\alpha$ , so  $q = 0$ .  $\square$

In view of Theorem 7.6 (v), Theorem 7.10 has the following corollary.

**7.11 Corollary.**  $p_U = 1$

In the same sense as Theorem 7.10, the eigenprojection  $d_\alpha$  may be interpreted as the “jump” that occurs as  $\lambda$  approaches  $\alpha$  from the left.

**7.12 Theorem.** *If  $\alpha \in \mathbb{R}$ , then  $p_\alpha - d_\alpha$  is the supremum in the OML  $P$  of  $B := \{p_\mu : \alpha > \mu \in \mathbb{R}\}$ .*

*Proof.* By Theorem 7.6 (iv),  $d_\alpha \leq p_\alpha$ , so  $p_\alpha - d_\alpha = p_\alpha \wedge (1 - d_\alpha) \in P$  by Lemma 4.8 (ii). Let  $(q_\lambda)_{\lambda \in \mathbb{R}}$  be the spectral resolution of  $-g$ . By Theorem 7.10 and Lemma 7.5,  $q_{-\alpha} = (1 - p_\alpha) + d_\alpha$  is the infimum in  $P$  of

$$\{q_\lambda : -\alpha < \lambda\} = \{(1 - p_{-\lambda} + d_{-\lambda} : -\lambda < \alpha\} = \{(1 - p_\mu) + d_\mu : \mu < \alpha\};$$

hence by duality in  $P$  (in this case, the De Morgan law),  $1 - ((1 - p_\alpha) + d_\alpha) = p_\alpha - d_\alpha$  is the supremum in  $P$  of

$$C := \{1 - ((1 - p_\mu) + d_\mu) : \mu < \alpha\} = \{p_\mu - d_\mu : \mu < \alpha\}.$$

We have to show that  $p_\alpha - d_\alpha = p_\alpha \wedge (1 - d_\alpha)$  is also the supremum in  $P$  of  $B$ . If  $\mu < \alpha$ , then by parts (iii) and (iv) of Theorem 7.6,  $p_\mu \leq p_\alpha \wedge (1 - d_\alpha)$ , i.e.,  $p_\alpha \wedge (1 - d_\alpha)$  is an upper bound for  $B$ . Suppose that  $r \in P$  is another upper bound for  $B$ . Then, if  $\mu < \alpha$ , we have  $p_\mu - d_\mu \leq p_\mu \leq r$ , i.e.,  $r$  is an upper bound for  $C$ ; hence  $p_\alpha - d_\alpha \leq r$ , so  $p_\alpha - d_\alpha$  is the supremum of  $B$ .  $\square$

## 8 Blocks and C-blocks

According to Theorem 4.5,  $P$  is a  $\sigma$ -complete orthomodular lattice. Recall that elements  $p, q \in P$  are called (*Mackey*) *compatible* iff there are pairwise orthogonal elements  $p_1, q_1, r \in P$  such that  $p = p_1 \vee r = p_1 + r$ ,  $q = q_1 \vee r = q_1 + r$ .

**8.1 Lemma.** *Two elements  $p, q \in P$  are compatible iff they commute.*

*Proof.* Let  $pq = qp$  and put  $r = pq$ . By Lemma 4.8 (iv),  $pq = p \wedge q \in P$ . Then  $p = (p - r) + r$ ,  $q = (q - r) + r$ , and  $r, p - r, q - r$  are pairwise orthogonal.

Conversely, let  $p = p_1 + r$ ,  $q = q_1 + r$  with  $p_1, q_1, r$  pairwise orthogonal. Then  $pq = qp = r$ .  $\square$

A subset  $B$  of  $P$  is called a *block* of  $P$  if  $B$  is a maximal set of pairwise compatible elements [14, Ch.1, §4]. In view of Lemma 8.1, it is clear that  $B \subseteq P$  is a block of  $P$  iff  $B = C(B) \cap P$ . It is well known that every block in  $P$  is a maximal Boolean  $\sigma$ -subalgebra of  $P$ . Following [9, Def. 5.1], a subgroup of  $G$  having the form  $C(B)$ , where  $B$  is a block in  $P$ , will be called a *C-block* in  $G$ .

**8.2 Theorem.** *A subset  $H$  of  $G$  is a C-block of  $G$  iff  $H$  is a maximal set of pairwise commuting elements of  $G$ .*

*Proof.* If  $H \subseteq G$ , it is clear that  $H$  is a maximal set of pairwise commuting elements of  $G$  iff  $H = C(H)$ . Suppose  $H = C(B)$  for some block  $B = C(B) \cap P$  of  $P$ . Then by Corollary 7.9 (ii),  $H = C(B) = C(C(B) \cap P) = CC(B) = C(H)$ . Conversely, suppose  $H = C(H)$  and put  $B := H \cap P = C(H) \cap P$ . Then  $CC(H) = C(H) = H$  and, again by Corollary 7.9 (ii),  $B = H \cap P = CC(H) \cap P = C(C(H) \cap P) \cap P = C(B) \cap P$ .  $\square$

As a consequence of Theorem 8.2,  $G$  is covered by its own C-blocks. Moreover, as we proceed to show, each C-block  $H$  in  $G$  is itself an AH-algebra that has the structure of an archimedean lattice-ordered commutative real Banach algebra.

**8.3 Theorem.** *Let  $H$  be a C-block in  $G$ . Then  $(R, E \cap H)$  is an e-ring with directed group  $H$ , the e-ring partial order on  $H$  is the partial order induced from  $G$ , the set of projections in  $H$  is a block  $B$  in  $P$ ,  $H = C(B)$ , and  $H$  is an AH-algebra with the Vigier property. Moreover, under the 1-norm  $\|\cdot\|$ ,  $H$  is a commutative and associative real Banach algebra with unity element 1 and with the property that  $h \in H \Rightarrow \|h^2\| = \|h\|^2$ .*

*Proof.* By definition of a C-block, there exists a block  $B$  in  $P$  such that  $H = C(B)$ . We omit the straightforward verification that  $(R, E \cap H)$  satisfies the conditions in Definition 2.1, that  $H$  is the directed group of  $(R, E \cap H)$ , that  $B$  is the set of projections in  $H$ , and that the e-ring partial order on  $H$  is the restriction to  $H$  of the partial order on  $G$ . Obviously,  $\frac{1}{2} \in C(B) = H$  and  $H$  inherits the QA property from  $G$ .

To prove that  $H = C(B)$  has the V property, suppose  $h_1 \leq h_2 \leq \dots$  is an ascending sequence in  $H$  that is bounded above in  $H$ . Then the sequence is bounded above in  $G$ , and by Theorem 8.2, the elements of the sequence commute pairwise, hence by CV it has a supremum  $h$  in  $G$  and  $h \in CC\{h_n : n \in \mathbb{N}\}$ . If  $p \in B$ , then  $pCh_n$  for all  $n \in \mathbb{N}$ , and therefore  $hCp$ . Consequently,

$h \in C(B) = H$ , so  $h$  is the supremum of the sequence  $(h_n)_{n \in \mathbb{N}}$  in  $H$ , and  $h$  double commutes in  $H$  with the set  $\{h_n : n \in \mathbb{N}\}$ . Thus,  $H$  has the V property, hence it has the CV property, and therefore  $H$  is an AH-algebra.

Obviously,  $H = C(B)$  is closed under multiplication by real numbers, and if  $g, h \in H$ , then  $gh = hg \in G$  by Theorem 8.2 and Lemma 2.8 (i), whence  $gh \in H$ . Therefore,  $H$  is a commutative and associative real linear algebra with unity element 1. By Theorem 6.5 (iv),  $H$  is a normed linear algebra under the 1-norm. As  $H$  has the V property, it is monotone  $\sigma$ -complete, whence it is a Banach algebra under the 1-norm by Theorem 6.6. By Theorem 6.5 (ii),  $\|h^2\| = \|h\|^2$  for all  $h \in H$ .  $\square$

Let  $\mathcal{A}$  be a linear algebra over  $\mathbb{R}$ . We say that  $\mathcal{A}$  is a *partially ordered linear algebra* iff the additive group of  $\mathcal{A}$  is a partially ordered abelian group, and whenever  $0 \leq a, b \in \mathcal{A}$  and  $0 \leq \lambda \in \mathbb{R}$ , we have  $0 \leq ab$  and  $0 \leq \lambda a$ . If a partially ordered linear algebra  $\mathcal{A}$  is a lattice, it is called an  $\ell$ -algebra [11].

**8.4 Theorem.** *Let  $H$  be a  $C$ -block in  $P$ . Then: (i)  $h \in H \Rightarrow |h|, h^+, h^-, h^\circ \in H$ . (ii)  $H$  is an archimedean Dedekind  $\sigma$ -complete  $\ell$ -algebra with order unit 1. (iii) If  $g, h \in H$ , then the infimum and supremum of  $g$  and  $h$  in  $H$  are given by  $g \wedge_H h = g - (g - h)^+$  and  $g \vee_H h = g + (h - g)^+$ . (iv) If  $h \in H$ , then the spectral resolution and the family of eigenprojections of  $h$  are the same whether calculated in  $G$  or in  $H$ .*

*Proof.* There is a block  $B$  in  $P$  such that  $H = C(B)$ .

(i) If  $h \in H = C(B)$ , then  $|h|, h^+, h^- \in H$  by Lemma 4.10 (ii), and  $h^\circ \in H$  by Theorem 4.2.

(ii) Obviously,  $H = C(B)$  is an archimedean partially ordered algebra over  $\mathbb{R}$  and 1 is an order unit in  $H$ . To prove that  $H$  is a lattice, let  $g, h \in H$  and put  $p := ((g - h)^+)^{\circ}$ . Then by (i),  $p \in H \cap P = C(B) \cap P = B$ , and by Theorem 5.2,  $p \in P^\pm(g - h)$ , so  $(1 - p)(g - h) \leq 0 \leq p(g - h)$  with  $(1 - p)(g - h), p(g - h) \in H$ . Put  $a := ph + (1 - p)g$ . Then  $a \in H$  and  $a \leq g, h$ . Suppose  $b \in H$  and  $b \leq g, h$ . Then  $pb \leq ph$  and  $(1 - p)b \leq (1 - p)g$ , so  $b \leq pb + (1 - p)b \leq a$ . Thus  $a$  is the infimum of  $g$  and  $h$  in  $H$ . The existence of the supremum of  $g$  and  $h$  in  $H$  is shown dually, hence  $H$  is a lattice. By Theorem 8.3,  $H$  has the V property, therefore it is monotone  $\sigma$ -complete, and consequently it is Dedekind  $\sigma$ -complete by [12, Lemma 16.7].

(iii) Let  $g, h \in H$ . Recall that in a comparability group, the pseudo-meet  $g \sqcap h$  and pseudo-join  $g \sqcup h$  are defined by  $g \sqcap h := g - (g - h)^+$ ,  $g \sqcup h := g + (h - g)^+$



[6, Definition 5.2]. By (i),  $g \sqcap h, g \sqcup h \in H$ , and by [6, Theorem 5.4 (iv)]  $g \wedge_H h = g \sqcap h$  and  $g \vee_H h = g \sqcup h$ .

(iv) Follows directly from (i) and Definition 7.3.  $\square$

In view of Theorem 8.4 (iii), we have the following.

**8.5 Corollary.** *Suppose that  $H_1$  and  $H_2$  are C-blocks in  $G$  and that  $g, h \in H_1 \cap H_2$ . Then the infimum and supremum of  $g$  and  $h$  as calculated in  $H_1$  are the same as the infimum and supremum of  $g$  and  $h$  as calculated in  $H_2$ .*

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